Block hybrid methods for solving dynamical systems

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Hybrid block methods

What are hybrid block methods(HBM)

- Multi-step numerical methods for solving IVPs
- Related to the Implicit Runge-Kutta methods
- Derived using collocation on intra-step grid points
- Accuracy order can be progressively increased
- Can be used to solve IVPs, BVPs, PDEs

Problem formulation (IVPs)

General Goal

Find a continuous approximation to the solutions of

$$(1)\quad \dot{y}(t) = f(t, y), \quad y(t_0) = y_0,$$

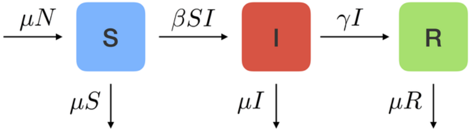
$$(2)\quad \ddot{y}(t) = f(t, y, \dot{y}), \quad y(t_0) = y_0,\ \dot{y}(t_0) = \delta_0,$$

$$(3)\quad \dot{y}(t) = f(t, y, y', y'')$$

$$(4)\quad \ddot{y}(t) = f(t, y, \dot{y}, y', y''),$$

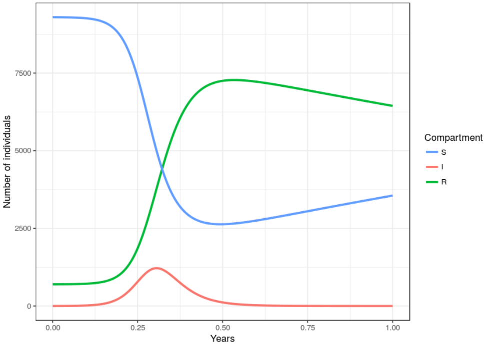
$$0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T$$

Sample Dynamical System: Mathematical Biology



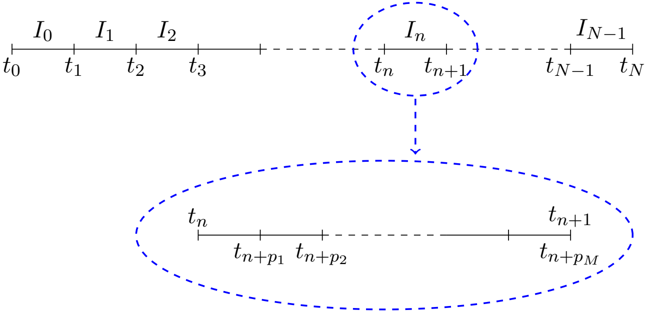
(Source:https://sineadmorris.github.io/post/the-sir-model/)

$$\frac{dS}{dt} = \mu N - \beta SI - \mu S$$
$$\frac{dI}{dt} = \beta SI - \gamma I - \mu I$$
$$\frac{dR}{dt} = \gamma I - \mu R.$$



Discretization (IVPs)

$$y'(t) = f(t, y), \;\; y(t_0) = y_0, \;\; t \in [t_0, t_N]$$



Collocation points

$$t_{n+p_i} = t_n + hp_i, \quad i = 1, 2, \ldots, M$$

Derivation of the method

Consider the IVP

$$y' = f(t, y), \quad y(t_0) = y_0$$

1. Set $y(t) \approx Y(t) = \sum_{k=0}^{m} c_k (t - t_n)^k$
2. Substitute in the IVP
3. Define $t_{n+p_i} = t_n + hp_i$ (Collocation points)
4. Apply initial condition

$$Y(t_0) = y_0$$

5. Collocate at t_{n+p_i} for $i = 1, 2, \ldots, m$ i.e

$$Y'(t_{n+p_i}) = f_{n+p_i}$$

Example with $\{0, \frac{1}{2}, 1\}$

```
M = 2;
points = Table[p_i = i/M, {i, 0, M}];
Table[t_(n+p_i) = (n + p_i) * h, {i, 0, M}];
Y = Sum[c_k * (t - t_n)^k, {k, 0, M + 1}];
equations = Table[(D[Y, t] /. t → t_(n+p_i)) == f_(n+p_i), {i, 0, M}] // Simplify;
unknowns = Table[c_i, {i, 0, M + 1}];
initial = (Y /. t → t_n) == y_n;
Allequations = Join[equations, {initial}];
solutions = Solve[Allequations, unknowns] // Flatten
yApprox = Y /. solutions;
equations = Table[y_(n+p_i) == yApprox /. t → t_(n+p_i), {i, 1, M}] // Expand;
Collect[equations, h, Together]
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$$\left\{c_0 \to y_n,\ c_1 \to f_n,\ c_2 \to -\frac{3 f_n - 4 f_{\frac{1}{2}+n} + f_{1+n}}{2 h},\ c_3 \to \frac{2\left(f_n - 2 f_{\frac{1}{2}+n} + f_{1+n}\right)}{3 h^2}\right\}$$

$$\left\{y_{\frac{1}{2}+n} == \frac{1}{24} h \left(5 f_n + 8 f_{\frac{1}{2}+n} - f_{1+n}\right) + y_n,\ y_{1+n} == \frac{1}{6} h \left(f_n + 4 f_{\frac{1}{2}+n} + f_{1+n}\right) + y_n\right\}$$

General form of the HBM

HBM

$$y_{n+p_i} = y_n + h\sum_{j=0}^{m} \beta_{i,j} f_{n+p_j}, \quad \beta_{i,j} = \int_0^{p_i} \ell_j(\tau)d\tau$$

$$i = 1, 2, \ldots, m$$

Matrix Form $\quad A_1 Y_{n+1} = A_0 Y_n + h(B_0 F_n + B_1 F_{n+1})$

HBM Examples (equally-spaced points)

$$\boxed{\text{Matrix Form} \qquad A_1 Y_{n+1} = A_0 Y_n + h(B_0 F_n + B_1 F_{n+1})}$$

For example when $M = 2$, we have

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},\ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ B_0 = \begin{pmatrix} 0 & \frac{5}{24} \\ 0 & \frac{1}{6} \end{pmatrix}, B_1 = \begin{pmatrix} \frac{1}{3} & -\frac{1}{24} \\ \frac{2}{3} & \frac{1}{6} \end{pmatrix}$$

$$F_n = \left[f_{n-\frac{1}{2}}, f_n\right], F_{n+1} = \left[f_{n+\frac{1}{2}}, f_{n+1}\right], Y_n = \left[y_{n-\frac{1}{2}}, y_n\right], Y_{n+1} = \left[y_{n+\frac{1}{2}}, y_{n+1}\right]$$

when $M = 3$ we have

$$A_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ B_0 = \begin{pmatrix} 0 & 0 & \frac{1}{8} \\ 0 & 0 & \frac{1}{9} \\ 0 & 0 & \frac{1}{8} \end{pmatrix}$$

$$B_1 = \begin{pmatrix} \frac{19}{72} & -\frac{5}{72} & \frac{1}{72} \\ \frac{4}{9} & \frac{1}{9} & 0 \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix}, \qquad \begin{aligned} F_n &= \left[f_{n-\frac{2}{3}}, f_{n-\frac{1}{3}}, f_n\right], \\ F_{n+1} &= \left[f_{n+\frac{1}{3}}, f_{n+\frac{2}{3}}, f_{n+1}\right], \\ Y_n &= \left[y_{n-\frac{2}{2}}, y_{n-\frac{1}{3}}, y_n\right], \\ Y_{n+1} &= \left[y_{n+\frac{1}{3}}, y_{n+\frac{2}{3}}, y_{n+1}\right] \end{aligned}$$

Numerical example

Example

$$y' = 2y, \qquad y(0) = 1$$

Exact solution is $y(t) = e^{2t}$

In this example, $f(t, y) = 2y$, thus $F_n = 2Y_n, \quad F_{n+1} = 2Y_{n+1}$ and

$$A_1 Y_{n+1} = A_0 Y_n + h(B_0 F_n + B_1 F_{n+1})$$
$$A_1 Y_{n+1} = A_0 Y_n + h(2B_0 Y_n + 2B_1 Y_{n+1})$$

$$\boxed{Y_{n+1} = \underbrace{(A_1 - 2hB_1)^{-1}(A_0 + 2hB_0)}_{\phi(h)} Y_n} \tag{1}$$

Results for $y' = 2y$

When $M = 2$

$$Y_{n+1} = \phi(h) Y_n = \begin{pmatrix} 0 & \frac{6-h^2}{2h^2-6h+6} \\ 0 & \frac{h^2+3h+3}{h^2-3h+3} \end{pmatrix} Y_n$$

When $M = 3$

$$Y_{n+1} = \phi(h) Y_n = \begin{pmatrix} 0 & 0 & \frac{2h^3-3h^2-27h+81}{-6h^3+33h^2-81h+81} \\ 0 & 0 & \frac{-2h^3-3h^2+27h+81}{-6h^3+33h^2-81h+81} \\ 0 & 0 & \frac{2h^3+11h^2+27h+27}{-2h^3+11h^2-27h+27} \end{pmatrix} Y_n$$