Block hybrid methods for solving dynamical systems - Numerical Experimentation

S.S. Motsa

University of Eswatini

sandilemotsa@gmail.com

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General form of the HBM

HBM

$$y_{n+p_i} = y_n + h\sum_{j=0}^{m} \beta_{i,j} f_{n+p_j}, \quad \beta_{i,j} = \int_0^{p_i} \ell_j(\tau)d\tau$$

$$i = 1, 2, \ldots, m$$

Matrix Form $\quad A_1 Y_{n+1} = A_0 Y_n + h(B_0 F_n + B_1 F_{n+1})$

Sample HBM

Matrix Form $A_1 Y_{n+1} = A_0 Y_n + h(B_0 F_n + B_1 F_{n+1})$

Grid points A: $\left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$

$$B_0 = \begin{pmatrix} 0 & 0 & \frac{1}{8} \\ 0 & 0 & \frac{1}{9} \\ 0 & 0 & \frac{1}{8} \end{pmatrix}, Y_{n+1} = \begin{bmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \end{bmatrix}$$

$$B_1 = \begin{pmatrix} \frac{19}{72} & -\frac{5}{72} & \frac{1}{72} \\ \frac{4}{9} & \frac{1}{9} & 0 \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix}$$

Grid points B: $\left\{0, \frac{1}{6}, \frac{1}{3}, 1\right\}$

$$B_0 = \begin{pmatrix} 0 & 0 & \frac{19}{288} \\ 0 & 0 & \frac{1}{18} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, Y_{n+1} = \begin{bmatrix} y_{n+\frac{1}{6}} \\ y_{n+\frac{1}{3}} \\ y_{n+1} \end{bmatrix}$$

$$B_1 = \begin{pmatrix} \frac{43}{360} & -\frac{11}{576} & \frac{1}{2880} \\ \frac{2}{9} & \frac{1}{18} & 0 \\ -\frac{6}{5} & \frac{3}{2} & \frac{1}{5} \end{pmatrix}$$

Numerical Experimentation

General autonomous systems

$$y' = f(t, y) = cy + d$$

$$F_{n+1} = \begin{bmatrix} f\left(t_{n+\frac{1}{3}}, y_{n+\frac{1}{3}}\right) \\ f\left(t_{n+\frac{2}{3}}, y_{n+\frac{2}{3}}\right) \\ f\left(t_{n+1}, y_{n+1}\right) \end{bmatrix} = \begin{bmatrix} cy_{n+\frac{1}{3}} + d \\ cy_{n+\frac{2}{3}} + d \\ cy_{n+1} + d \end{bmatrix} = c \underbrace{\begin{bmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \end{bmatrix}}_{Y_{n+1}} + \underbrace{\begin{bmatrix} d \\ d \\ d \end{bmatrix}}_{\mathbf{d}} = cY_{n+1} + \mathbf{d}$$

$$\begin{aligned} A_1 Y_{n+1} &= A_0 Y_n + h(B_0 F_n + B_1 F_{n+1}) \\ A_1 Y_{n+1} &= A_0 Y_n + hB_0(cY_n + \mathbf{d}) + hB_1(cY_{n+1} + \mathbf{d}) \\ Y_{n+1} &= PY_n + Q \end{aligned}$$

where $\quad P = (A_1 - chB_1)^{-1}(A_0 + chB_0), \quad Q = h(A_1 - chB_1)^{-1}(B_0 + B_1)\mathbf{d}$

Numerical Experimentation

General non-autonomous systems

$$y' = f(t,y) = c(t)y + d(t)$$

$$F_{n+1} = \begin{bmatrix} f\left(t_{n+\frac{1}{3}}, y_{n+\frac{1}{3}}\right) \\ f\left(t_{n+\frac{2}{3}}, y_{n+\frac{2}{3}}\right) \\ f\left(t_{n+1}, y_{n+1}\right) \end{bmatrix} = \begin{bmatrix} c\left(t_{n+\frac{1}{3}}\right) y_{n+\frac{1}{3}} + d\left(t_{n+\frac{1}{3}}\right) \\ c\left(t_{n+\frac{2}{3}}\right) y_{n+\frac{2}{3}} + d\left(t_{n+\frac{2}{3}}\right) \\ c\left(t_{n+1}\right) y_{n+1} + d\left(t_{n+1}\right) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} c\left(t_{n+\frac{1}{3}}\right) & 0 & 0 \\ 0 & c\left(t_{n+\frac{2}{3}}\right) & 0 \\ 0 & 0 & c\left(t_{n+1}\right) \end{bmatrix}}_{c_{n+1}} \underbrace{\begin{bmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \end{bmatrix}}_{Y_{n+1}} + \underbrace{\begin{bmatrix} d\left(t_{n+\frac{1}{3}}\right) \\ d\left(t_{n+\frac{2}{3}}\right) \\ d\left(t_{n+1}\right) \end{bmatrix}}_{d_{n+1}}$$

$$\begin{aligned} A_1 Y_{n+1} &= A_0 Y_n + h(B_0 F_n + B_1 F_{n+1}) \\ Y_{n+1} &= P_n Y_n + Q_n \end{aligned}$$

where $P_n = (A_1 - hB_1 c_{n+1})^{-1}(A_0 + hB_0 c_n)$, $Q_n = h(A_1 - hB_1 c_{n+1})^{-1}(B_0 d_n + B_1 d_{n+1})$

Numerical Experimentation

Autonomous system (time-invariant)

$$\frac{dy}{dt} = 2y + 4, \quad y(0) = 1, \quad \text{Exact: } y(t) = 3e^{2t} - 2$$

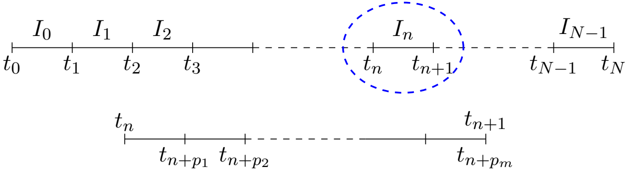
$c = 2$ and $d = 4$ and $Y_{n+1} = PY_n + Q$

$$\text{Points A: } \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}, \quad P = \begin{pmatrix} 0 & 0 & \frac{9784}{9153} \\ 0 & 0 & \frac{20917}{18306} \\ 0 & 0 & \frac{7453}{6102} \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1262}{9153} \\ \frac{2611}{9153} \\ \frac{1351}{3051} \end{pmatrix}$$

$$\text{Points B: } \left\{0, \frac{1}{6}, \frac{1}{3}, 1\right\}, \quad P = \begin{pmatrix} 0 & 0 & \frac{620945}{600588} \\ 0 & 0 & \frac{160498}{150147} \\ 0 & 0 & \frac{61130}{50049} \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{20357}{300294} \\ \frac{20702}{150147} \\ \frac{22162}{50049} \end{pmatrix}$$

Matrix structure

$$y_{n+p_i} = y_n + hp_i, \;\; t_{n+p_i} = t_n + hp_i, \;\; i = 1, \dots, m$$



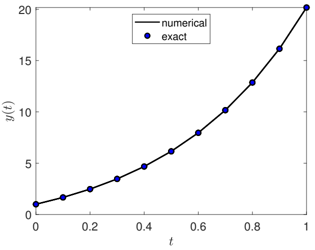
$$T = \begin{bmatrix} t_0 & t_{p_1} & t_{1+p_1} & t_{2+p_1} & \cdots & t_{N-1+p_1} \\ t_0 & t_{p_2} & t_{1+p_2} & t_{2+p_2} & \cdots & t_{N-1+p_2} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ t_0 & t_{p_m} & t_{1+p_m} & t_{2+p_m} & \cdots & t_{N-1+p_m} \end{bmatrix}$$

$$Y = \begin{bmatrix} y_0 & y_{p_1} & y_{1+p_1} & y_{2+p_1} & \cdots & y_{N-1+p_1} \\ y_0 & y_{p_2} & y_{1+p_2} & y_{2+p_2} & \cdots & y_{N-1+p_2} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ y_0 & y_{p_m} & y_{1+p_m} & y_{2+p_m} & \cdots & y_{N-1+p_m} \end{bmatrix}$$

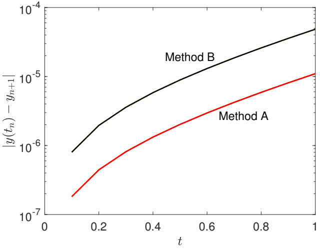
Results: Autonomous system example

Autonomous system (time-invariant)

$$\frac{dy}{dt} = 2y + 4, \quad y(0) = 1, \quad \text{Exact: } y(t) = 3e^{2t} - 2$$



(a) HBM Vs Exact solution

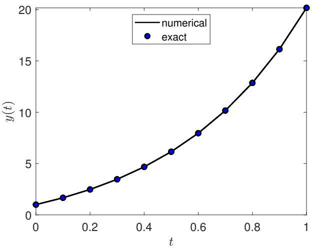


(b) Error profiles

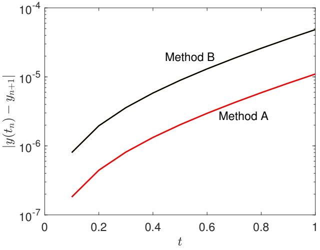
Results: Non-autonomous system example

Non-autonomous system

$$\frac{dy}{dt} = 2ty + 4t, \quad y(0) = 1, \quad \text{Exact: } y(t) = 3e^{t^2} - 2$$



(a) HBM Vs Exact solution



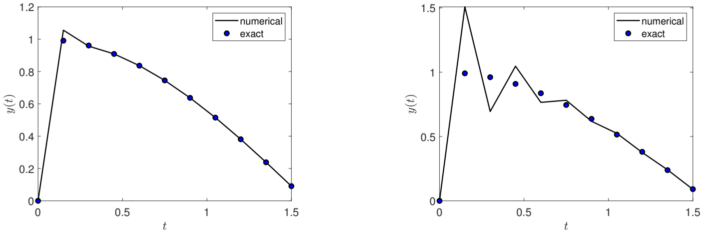
(b) Error profiles

Results: Stiff example

Non-autonomous system (stiff equation)

$$\frac{dy}{dt} = -50(y - \cos t), \quad y(0) = 0,$$

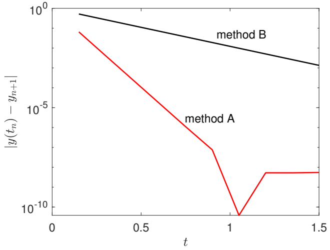
$$\text{Exact: } y(t) = \frac{50\left(-50e^{-50t} + \sin(t) + 50\cos(t)\right)}{2501}$$



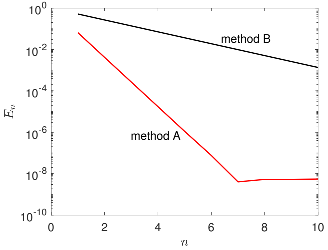
(a) Method A (b) Method B

Figure: Solution Profile

Results: Stiff example



(a) Solution Error



(b) Max Error

$$E_n = |y(t_{n+p_i}) - y_{n+p_i}|_\infty, \quad i = 1, 2, \ldots, m$$

Example from Chaos Theory

Consider the Lorenz system given by

$$
\begin{aligned}
\dot{y_1} &= a(y_2 - y_1), & y_1(0) &= 1\\
\dot{y_2} &= -y_1 y_3 + b y_1 - y_2, & y_2(0) &= 5\\
\dot{y_3} &= y_1 y_2 - c y_3, & y_3(0) &= 10
\end{aligned}
$$

The parameters of the iteration scheme are

$$
\begin{aligned}
c_1(t, y_1, y_2, y_3) &= -a, \quad & d_1(t, y_1, y_2, y_3) &= a y_2,\\
c_2(t, y_1, y_2, y_3) &= -1, \quad & d_2(t, y_1, y_2, y_3) &= b y_1 - y_1 y_3,\\
c_3(t, y_1, y_2, y_3) &= -c, \quad & d_3(t, y_1, y_2, y_3) &= y_1 y_2
\end{aligned}
$$

c_1, c_2 and c_3 are coefficients of y_1, y_2 and y_2, respectively

Sample Matlab code: Lorenz equation

```
a = 10; b = 28; c = 8/3;
c1 = @(t,y1,y2,y3)-a;
d1 = @(t,y1,y2,y3)a*y2;
c2 = @(t,y1,y2,y3) -1;
d2 = @(t,y1,y2,y3)b*y1 - y1*y3;
c3 = @(t,y1,y2,y3)-c;
d3 = @(t,y1,y2,y3)y1*y2  ;

Nt = 4000; t0 = 0; tT = 40;
h = (tT - t0)/Nt;
t = linspace(t0,tT,Nt+1);
 %initial conditions
y10 = 1; y20 = 5; y30 = 10;
```

Figure: Time series solution of the Lorenz equation

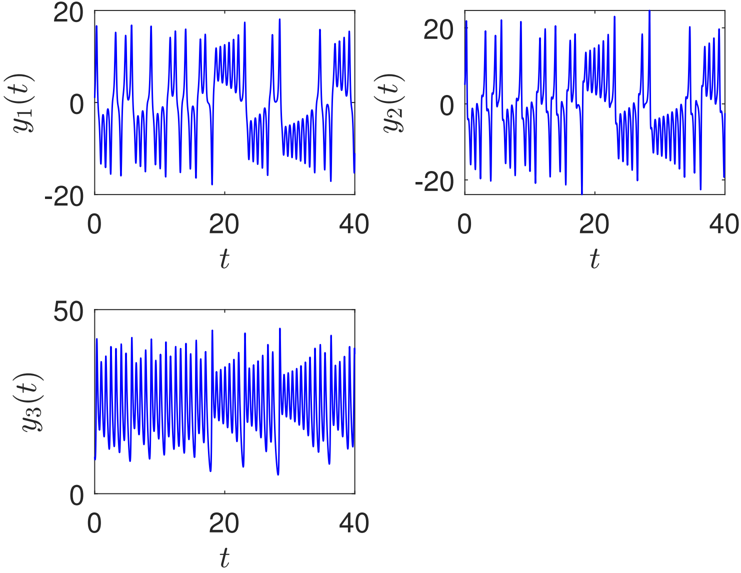


Figure: Phase portraits of the Lorenz equation