Block hybrid methods for solving dynamical systems - Stability

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Namibian International Spring School in Mathematics, University of Namibia, Windhoek, October 23-28, 2022

General form of the HBM

HBM

$$y_{n+p_i} = y_n + h\sum_{j=0}^{m} \beta_{i,j} f_{n+p_j}, \quad \beta_{i,j} = \int_0^{p_i} \ell_j(\tau)d\tau$$

$$i = 1, 2, \dots, m$$

Matrix Form $\qquad A_1 Y_{n+1} = A_0 Y_n + h(B_0 F_n + B_1 F_{n+1})$

Zero-Stability

Definition of zero-stability

- Roots of characteristic polynomial $\rho(\lambda)$ satisfy $|\lambda_j| \leq 1$.
- Roots with $|\lambda_j| = 1$ have multiplicity of 1.

$$\rho(\lambda) = det\,[A_1\lambda - A_0]$$

For HBMs, we have

$$A_0 = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$\rho(\lambda) = \lambda^m(\lambda - 1) \implies \text{HBM is zero-stable}$$

Truncation error

$$\text{Given} \quad y_{n+p_i} = y_n + h\sum_{j=0}^{m} \beta_{i,j} f_{n+p_j},$$

$$\text{Write} \quad \mathcal{L}_i[z(t_n);h] = z(t_n + hp_i) - z(t_n) - h\sum_{j=0}^{m} \beta_{i,j} z'(t_n + hp_j) \quad (1)$$

Apply Taylor series to expand (1) gives

$$\begin{aligned}
\mathcal{L}_i[z(t_n);h] &= \sum_{k=1}^{K} \frac{p_i^k}{k!} h^k z^{(k)}(t_n) - \sum_{k=1}^{K} \frac{h^k k}{k!} \sum_{j=0}^{m} \beta_{i,j} p_j^{k-1} z^{(k)}(t_n) + O(h^{K+1}) \\
&= \sum_{k=2}^{m+1} \frac{h^k}{k!} \left[p_i^k - k\sum_{j=0}^{m} \beta_{i,j} p_j^{k-1} \right] z^{(k)}(t_n) \\
&+ \sum_{k=m+2}^{K} \frac{h^k}{k!} \left[p_i^k - k\sum_{j=0}^{m} \beta_{i,j} p_j^{k-1} \right] z^{(k)}(t_n) + \mathcal{O}(h^{K+1})
\end{aligned}$$

Truncation error

Use the simplifying assumption

$$\sum_{j=0}^{m} \beta_{i,j} p_j^{k-1} = \frac{p_i^k}{k}, \quad \text{(due to J.C. Butcher in [1])}$$

gives

$$\mathcal{L}_i[z(t_n);h] = \sum_{k=m+2}^{K} \frac{h^k}{k!} \left[p_i^k - k \sum_{j=0}^{m} \beta_{i,j} p_j^{k-1} \right] z^{(k)}(t_n) + \mathcal{O}(h^{K+1})$$

Thus,

$$\boxed{\mathcal{L}_i[z(t_n);h] = \frac{h^{m+2}}{(m+2)!} \left[p_i^{m+2} - (m+2) \sum_{j=0}^{m} \beta_{i,j} p_j^{m+1} \right] z^{(m+2)}(t_n) + \mathcal{O}(h^{m+3})}$$

Truncation error examples for $M = 3$

Grid points A: $\left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$

$$\mathcal{L}_1[z(t_n); h] = -\frac{19h^5 y^{(5)}(t_n)}{174960} + O\left(h^6\right),$$
$$\mathcal{L}_2[z(t_n); h] = -\frac{h^5 y^{(5)}(t_n)}{21870} + O\left(h^6\right),$$
$$\mathcal{L}_3[z(t_n); h] = -\frac{h^5 y^{(5)}(t_n)}{6480} + O\left(h^6\right)$$

Grid points B: $\left\{0, \frac{1}{6}, \frac{1}{3}, 1\right\}$

$$\mathcal{L}_1[z(t_n); h] = -\frac{83h^5 y^{(5)}(t_n)}{11197440} + O\left(h^6\right),$$
$$\mathcal{L}_2[z(t_n); h] = -\frac{h^5 y^{(5)}(t_n)}{699840} + O\left(h^6\right),$$
$$\mathcal{L}_3[z(t_n); h] = -\frac{19h^5 y^{(5)}(t_n)}{25920} + O\left(h^6\right)$$

Both sets of points give a HBM method that appears to have a high order of accuracy and is consistent.

Linear stability (Absolute stability)

Region of absolute stability(RAS)

The RAS is considered to be the set of points $z \in \mathbb{C}$ such that the roots of the characteristic equation, associated with the Dahlquist test equation

$$y' = \lambda y$$

lie within the unit circle.

When applied to the HBM with grid points A and B, we get

$$Y_{n+1} = R(z)Y_n, \;\text{ where }\; R(z) = (A_1 - zB_1)^{-1}(A_0 + zB_0). \tag{2}$$

The stability function $R(z)$ is defined as

$$R(z) = \frac{z^3 + 11z^2 + 54z + 108}{-z^3 + 11z^2 - 54z + 108}, \qquad \text{Points A}$$

$$R(z) = -\frac{2\left(5z^3 + 37z^2 + 135z + 216\right)}{z^3 - 20z^2 + 162z - 432}, \quad \text{Points B}$$

Instability appears if for an eigenvalue λ the modulus $|R(z)| > 1$.

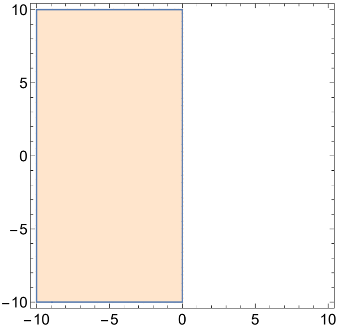
Absolute stability

Definition of A-stable

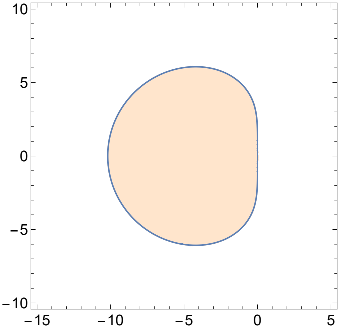
A method is A-stable is the stability domain

$$S := \{z : |R(z) \leq 1|\}$$

covers the entire left half plane $\mathbb{C}^-$.



(a) Grid points A (A-Stable)



(b) Grid points B (NOT A-Stable)

A-Stability through Order Stars

More insight is gained from comparing $|R(z)|$ to $|e^z|$

Definition of Order Stars

The order star of R is the region in the plane bordered by the curve(s) matching the condition

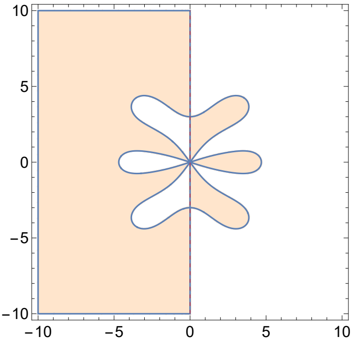
$$A := \{z \in \mathbb{C} : |R(z)| > |e^z|\}$$

They provide vital information, such as order and stability, in a unified structure.

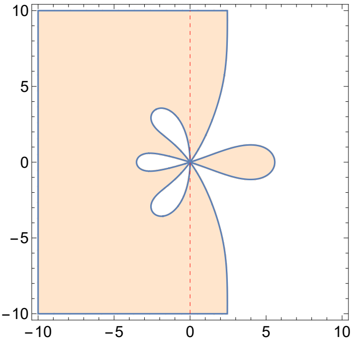
Conditions for A-stability [2]

- If the order star contains no portion of the imaginary axis and all poles remain within the right half plane
- A-stability fails if the order star covers a portion of the imaginary axis.

Order star graphs



(a) Grid points A (A-Stable)



(b) Grid points B (NOT A-Stable)

References

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- G. Wanner, E.Hairer, and S.P. Norsett, Order stars and stability theorems. BIT Numerical Mathematics, 18(4), 1978 pp.475-489.
- Hairer, E., Wanner, G. (2015). Order Stars and Stability Domains. In: Engquist, B. (eds) Encyclopedia of Applied and Computational Mathematics. Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-540-70529-1_134