Presenting frames - Part 2

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Abstract

This talk is a 3-part lecture series in which we present frames as distributive lattices satisfying the so-called infinite distibutive law. On one hand frames are viewed as Heyting algebras, on the other as generalized lattices of "opens". The latter view enables one to revisit many classical results of general topology - an exercise dubbed as "doing topology without points", "pointfree topology" or "pointless topology" - with the benefit, sometimes, of not having to rely heavily on choice principles.

Key words: complete lattice, frame, locale, sober space, spatial locale, sublocale.

We draw notions from topology, lattice theory and category theory.

Part 2

- Frames
- Sober spaces

Frames

A frame $(L, \wedge, \bigvee, 0, 1)$ is a complete lattice satisfying the infinite distributive law:

- $a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$, for any $a \in L$ and $S \subseteq L$.

Typical example of a frame:

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Typical example of a frame:

- Any topology $\mathcal{O}(X)$ on a set X.

Categorical view

We have the category **Top** of topological spaces and continuous maps.

$$X \xrightarrow{f} Y$$

We also have the category **Frm**

$$L \xrightarrow{h} M$$

h - frame map, which preserves $\wedge, \bigvee, 0, 1$

Frame map properties

Frame map properties

- The map $h_* : M \longrightarrow L$, defined by

$$h_*(a) = \bigvee \{x \in L \mid h(x) \leq a\}$$

is called the *right adjoint* of a frame homomorphism $h : L \longrightarrow M$.

Classical example from topology

- Continuous maps in **Top** translate to frame maps in **Frm**.

Top	**Frm**
X	$\mathcal{O}(X)$
$\uparrow f$	$\downarrow f^{-1}$
Y	$\mathcal{O}(Y)$

Categorical view

Dual category to **Frm**	$=$	the category **Loc** of locales
	$\equiv$	simply turn around the arrows in **Frm**

Categorical view

Conceptually...

- **Loc** - doing topology with generalized spaces.
- **Frm** - lattice theory applied to topology.

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- (ii) We say a is *complemented* if $a \vee a^* = 1$. In that case $a^* = \neg a$, the (full) complement of a.

General literature on frames/locales

[1]. Johnstone P.T., *Stone Spaces*, Cambridge Univ. Press, Cambridge, 1982.

[2]. Picado J. and Pultr A., *Frames and Locales: topology without points*, Frontiers in Mathematics, Springer, Basel, 2012.

Irreducible closed

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- A closed subset F in a topological space X is said to be *irreducible* if it cannot be expressed as a union of two proper closed subsets of itself.
- Thus, $F = F_1 \cup F_2$ implies $F = F_1$ or $F = F_2$ for any closed $F_1, F_2 \subseteq X$.

Sober space

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- We write **Sb** for the category of sober spaces and continuous maps. It is a full subcategory of **Top**.
- Soberness (or sobriety) is a separation property between T_0-ness and Hausdorffness, but with no relation to the T_1-property.

Points of a locale

The notion of "point of a locale" evolved from that of point of a given topological space: $1 \xrightarrow{p} X$.
Notice that p is a continuous function, whose inverse image $p^{-1} : \mathcal{O}(X) \longrightarrow \{\emptyset, 1\}$ is a frame homomorphism.

Thus, by a *point of a locale* L is meant a locale map $p : 2 \longrightarrow L$, where $2 = \{0, 1\}$ is the so-called 2-element locale.

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- We write $\mathrm{pt}(L)$ for the set of all points of L.
- Notice therefore that, for any given $p \in \mathrm{pt}(L)$, we have the corresponding frame homomorphism $p^* : L \longrightarrow 2$.

Points of a locale

For each $a \in L$, where L is a locale, consider the set we shall write as:

$$\varphi_L^*(a) = \{p \in \mathrm{pt}(L) \mid p^*(a) = 1\}$$

It turns out that $\varphi_L^*[L] = \{\varphi_L^*(a) \mid a \in L\}$ is a frame, and we shall therefore assign it as the topology on $\mathrm{pt}(L)$ induced by this process of assigning points to the locale L.

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- Thus, $\mathrm{pt}(L)$ is in fact the space of points of the locale L.
- We have also naturally defined a map $\varphi_L^* : L \longrightarrow \mathcal{O}(\mathrm{pt}(L))$, which is in fact a frame homomorphism.
- The corresponding locale map $\varphi_L : \mathcal{O}(\mathrm{pt}(L)) \longrightarrow L$ is known as the *counit map* for L.

Soberification

It turns out that, given any topological space X, the induced space of "localic points" $\mathrm{pt}(\mathcal{O}(X))$ is a sober space, known as the soberification of X.

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- The (continuous) map $\xi_X : X \longrightarrow \mathrm{pt}(\mathcal{O}(X))$ defined by: for each $x \in X$,
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- A topological space X is therefore sober if and only if the unit map ξ_X is a *homeomorphism*.

Spatiality

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- Writing **Lsp** for the category of spatial locales, one has that the categories **Sb** and **Lsp** are equivalent via the *adjoint* pair

$$\mathcal{O} : \mathbf{Sb} \rightleftarrows \mathbf{Lsp} : \mathrm{pt}$$

Ongoing work

James J. Madden & Martin Mugochi. *Paralocalic groups*; Topology and Its Applications, **259**, (2019), 275 - 282.

Giovanni Marelli & Martin Mugochi. *Toposes as spaces*; Unpublished work (2022)

Tangi Unene

Thank You!