HiDEAs to work with

or

Higher Order Derivations on Exterior Algebras

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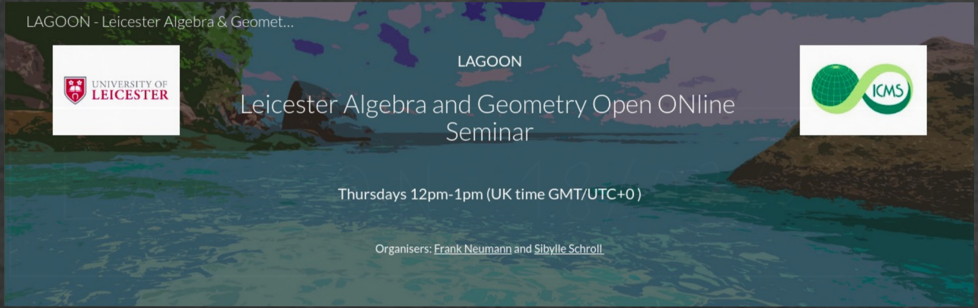


MARCH 18, 2021

- **1937**: Hasse & Schmidt define Higher Order Derivations for B-algebras R to provide substitutes for Taylor expansions of analytic functions (Crelle, **177**, 215–237)

$$\boxed{\underline{D} = (D_0, D_1, D_2, \dots) : R \longrightarrow R}$$

such that

$$\boxed{D_j(ab) = \sum_{i \geq 0} D_i a \cdot D_{j-i} b} \qquad \circledast$$

e.g.

$$\boxed{D_1(ab) = D_1 a \cdot b + a D_1 b, \quad D_2(ab) = D_2 a \cdot b + D_1 a \cdot D_1 b + a \cdot D_2 b, \quad \dots}$$

See e.g. Matsumura, Commutative Ring Theory, 1987, p. 207

Let $D(z) = \sum D_i z^i : R \longrightarrow R[[z]] \quad (D_i \in \mathrm{End}_B(R))$

Then Leibniz rules (*) can be summarised as :

$$\boxed{D(z)(ab) = D(z)a \cdot D(z)b}$$

If D_0 is invertible in $\mathrm{End}_B(R)$,

then $D(z)$ is invertible in $\mathrm{End}_B(R)[[z]]$,

i.e. $\exists\, \bar{D}(z) \in HS_B(R) \; / \; \boxed{D(z)\bar{D}(z) = \bar{D}(z)D(z) = 1}$

Writing $\bar{D}(z) = \sum_{i \geq 0} (-1)^i \bar{D}_i z^i$ one finds

$$\bar{D}_1 = D_1 \, , \qquad \bar{D}_2 = D_1^2 - D_2 \, , \quad \ldots$$

Proposition: $\bar{D}(z) : R \longrightarrow R[[z]]$ is a HS-derivation,

i.e. $\bar{D}(z)(a.b) = \bar{D}(z)a . \bar{D}(z)b$

In particular

$$\boxed{\bar{D}(z)a . b = \bar{D}(z)(a . D(z)b)}$$

(integration by parts)

If $D_0 = 1$ $\quad \bar{D}_1 a . b = \bar{D}_1(ab) - aD_1 b$ (integration by parts)

Many authors worked on HSD: P. Ribenboim, P. Vojta (for jet bundles), R. Skjelnes and one of the best active experts, Luis Narvaez (Sevilla).

HiDEA

or

Hasse-Schmidt Derivations on Exterior Algebra

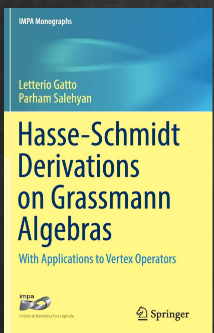
Let: $V_n := \dfrac{\mathbb{Q}[X]}{(X^n)}$ $\qquad \left(V := V_\infty = \mathbb{Q}[X] \right)$

Definition: a Hasse-Schmidt (HS) derivation on $\bigwedge V_n$ is a $\mathbb{Q}$-linear map:

$$\mathcal{D}(z) : \bigwedge V_n \longrightarrow \bigwedge V_n[[z]]$$

such that

$$\boxed{\mathcal{D}(z)(u \wedge v) = \mathcal{D}(z)u \wedge \mathcal{D}(z)v} \qquad \forall\, u, v \in \bigwedge V_n$$



Writing $\boxed{D(z) = \sum_{i\geq 0} D_i z^i}$ $\quad (D_i \in \mathrm{End}_{\mathbb{Q}}(\Lambda V_n))$

Higher Order Leibniz like rules hold

$$(D_0 + D_1 z + D_2 z^2 + \dots)(u \wedge v) = (D_0 u + D_1 u \cdot z + D_2 u \cdot z^2 + \dots) \wedge (D_0 v + D_1 v \cdot z + D_2 v \cdot z^2 + \dots)$$

From which

$$D_0(u \wedge v) = D_0 u \wedge D_0 v \;, \qquad D_1(u \wedge v) = D_1 u \wedge D_0 v + D_0 u \wedge D_1 v$$

and, in general, $\forall j \geq 0$

$$D_j(u \wedge v) = \sum_{i=0}^{j} D_i u \wedge D_{j-i} v$$

$$\boxed{\text{If } \quad D_0 = 1 \,, \qquad D_1(u \wedge v) = D_1 u \wedge v + u \wedge D_1 v}$$

Integration by Parts

Proposition. If $D_0 \in \mathrm{Aut}_{\mathbb{Q}}(V)$, then

$$\bar{D}(z) = \sum_{i \geq 0} (-1)^i \bar{D}_i z^i = \frac{1}{D(z)} \in \mathrm{End}_{\mathbb{Q}}(\wedge V_n)[[z]]$$

exists, is a HS derivation

and the integration by parts formula holds

$$\boxed{\bar{D}(z)(D(z)u \wedge v) = u \wedge \bar{D}(z)v} \qquad (*)$$

In particular

$$\boxed{D_1 u \wedge v = D_1(u \wedge v) - u \wedge D_1 v}$$

Exterior Powers $\Lambda^r \mathbb{Q}[X]$

Let $\mathcal{P} = \{ \underline{\lambda} = (\lambda_1 \geq \lambda_2 \geq \dots)$ all zero but finitely many $\}$ = partitions

$$|\underline{\lambda}| = \sum \lambda_j \qquad \ell(\underline{\lambda}) = \#\{ i \mid \lambda_i \neq 0 \}.$$

$\mathcal{P}_r$ = {partitions with at most r parts} $= \{ \underline{\lambda} \in \mathcal{P} \,/\, \lambda_j = 0 \text{ if } j \geq r+1 \}$

$$\Lambda^r V = \bigoplus_{\underline{\lambda} \in \mathcal{P}_r} \mathbb{Q} \cdot \underline{\lambda} \quad \text{usually written as} \quad \bigoplus \mathbb{Q}\, \underline{X}^r(\underline{\lambda})$$

where

$$\underline{X}^r(\underline{\lambda}) = X^{r-1+\lambda_1} \wedge X^{r-2+\lambda_2} \wedge \dots \wedge X^{\lambda_r}$$

$\left(\text{e.g. } \underline{X}^2(3,1) = X^{1+3} \wedge X^1 = X^4 \wedge X^1 \right)$

Exterior Algebra

$\Lambda V = \left(\bigoplus_{r \geq 0} \Lambda^r V, \wedge \right)$ with the supercommutative juxtaposition product "$\wedge$"

$\Lambda^k \mathbb{Q}[X] \cong \mathbb{Q}[e_1, e_2, \dots, e_k]$ $\quad$ $\Lambda^k V_n \cong H^*(G(k,n), \mathbb{Q})$

Let $B_r = \mathbb{Q}[e_1, e_2, \dots, e_r]$ be the ring of symmetric polynomials in r indeterminates. It is well known that

$$B_r = \bigoplus_{\underline{\lambda} \in P_r} \mathbb{Q} \cdot \Delta_{\underline{\lambda}}(H_r) \cong \Lambda^r \mathbb{Q}[X]$$

$$\Delta_{\underline{\lambda}}(H_r) = \det(h_{\lambda_j - j + i})$$

$$\left(\Delta_{(2,1)}(H_r) = \begin{vmatrix} h_2 & h_0 \\ h_3 & h_1 \end{vmatrix} \right)$$

example

Where $(h_j)_{j \in \mathbb{Z}}$ are the complete symmetric polinomials

$$\sum_{j \in \mathbb{Z}} h_j z^j = \frac{1}{1 - e_1 z + \dots + (-1)^r e_r z^r}$$

Applications

a) Basic Multilinear Algebra (e.g. Cayley-Hamilton Theorem) $\boxed{\begin{array}{l}(-,\ \text{Scherbak})\\(-,\ \text{Rowen})\end{array}}$

b) Schubert Calculus (Classical, Quantum, Equivariant) (–, 2005)

via Schubert Derivations; (–, Cordovez, Santiago, 2007, 2008)

c) Representation theory

- vertex operators, Boson-Fermion Correspondence (–, Salehyan 2014)
- KP Hierarchy, DJKM vertex operator representation of a_∞ (2019)

(–, Behzad, Contiero, Chapman, Rowen, Scherbak, Salehyan, Vidal Martins, ...)

MAIN EXAMPLES

a) Let $d(z) = \sum d_i z^i : \Lambda V_m \longrightarrow \Lambda V_m[[z]]$ a "derivation"

$$d(z)(u \wedge v) = d(z)u \wedge v + u \wedge d(z)v \quad \Big| \quad d(z)\Lambda^r V_m \subseteq \Lambda^r V_m[[z]]$$

Then $\underline{\exp(d(z)) : \Lambda V_m \longrightarrow \Lambda V_m[[z]]} \in HS(\Lambda V_m)$

b) Let $f \in gl(V_m) = \{$Lie algebra of endomorphisms of $V_m\}$

$$\delta : gl(V_m) \longrightarrow \mathrm{End}_{\mathbb{Q}}(\Lambda V_m)$$

$$f \longmapsto \delta(f)$$

such that

$$\boxed{\begin{aligned} &\delta(f)(u) = f(u), \qquad \forall u \in V_m \\ &\delta(f)(v \wedge w) = \delta(f)v \wedge w + v \wedge \delta(f)w, \quad \forall v, w \in \Lambda V_m \end{aligned}}$$

Proposition. $D^f(z) = \sum D_i^f z^i = \boxed{\exp\left(\sum_{i\geq 1} \frac{1}{i}\,\delta(f^i)\,z^i\right)}$

is the unique HS-derivation on ΛV_n such that

$$D^f(z)(u) = \sum_{n\geq 0} f^n(u)\, z^n \in V[[z]]$$

$$\boxed{\operatorname{Exp}(\delta(fz))}$$

$$\|$$

Its inverse is: $\bar{D}^f(z) = \sum_{i\geq 0} (-1)^i \bar{D}_i^f z^i = \boxed{\exp\left(-\sum_{i\geq 1} \frac{1}{i}\,\delta(f^i)\,z^i\right)}$

and

$$\bar{D}^f(z)u = \underline{u - f(u)z}$$

$$\|$$

$$= u - \bar{D}_1 u\cdot z + \bar{D}_2 u z^2 - \bar{D}_3 u\cdot z^3 + \dots$$

(In particular

$\bar{D}_i^f|_{V_n} = 0$ if $i > 1$ $\Rightarrow$ by induction $\bar{D}_i^f|_{\Lambda^k V_n} = 0$ if $i > k$)

Integration by parts for $D^f(z)$:

$$\boxed{\bar{D}^f(z)\left(D^f(z)\,u \wedge v\right) = u \wedge \bar{D}^f(z)\,v}$$

is the Cayley-Hamilton theorem.

Suppose $\dim V_n = n < \infty$. Then $\dim_{\mathbb{Q}} \Lambda^n V_n = 1$

Let $\bar{D}^f_i \xi = e_i(f)\,\xi$, $e_i(f) \in \mathbb{Q}$

traces

$$\boxed{\bar{D}^f(z)\xi = \underline{(1 - e_1(f)z + \dots + (-1)^n e_n(f) z^n)}\,\xi}$$

<u>Theorem</u> (–, <u>Scherbak</u>, 2019)

$$\boxed{D^f_n - e_1(f)\cdot D^f_{n-1} + \dots + (-1)^n e_n(f) = 0 \quad \text{in} \quad \boxed{\Lambda V_n}}$$

In particular, for all $\underline{u} \in V_n$:

$$D^f_n(u) - e_1(f)\, D^f_{n-1}(u) + \dots + (-1)^n e_n(f)\underline{u} = 0 \qquad (D^f_i u = f^i(u))$$

$$\Downarrow$$

$$(f^n - e_1 f^{n-1} + \dots + (-1)^n e_n)\underline{u} = 0 \qquad \forall\, \underline{u} \in V_n$$

Idea of proof

Let $u \in \Lambda^k V_n$. Then for all $\underline{v} \in \Lambda^{n-k} V_n$

$$D^f_n(u) - e_1 D^f_{n-1}(u) + \dots + (-1)^n e_n(f) u \wedge v = 0$$

In fact:

$$\bar{D}^f(z)(D^f(z) u \wedge v) = u \wedge \bar{D}^f(z) v$$

$$(1 - e_1 z + \dots + e_n z^n)(D^f(z) u \wedge v) = u \wedge \bar{D}^f(z) v$$

$$D^f_n u \wedge v - e_1 D^f_{n-1} u \wedge v + \dots + (-1)^n e_n D^f_0 u \wedge v = u \wedge \underset{\substack{\parallel \\ 0}}{\bar{D}^f_n} v \quad (n > n-k) \qquad \square$$

SCHUBERT DERIVATIONS

Recall: $V_n = \dfrac{\mathbb{Q}[X]}{(X^n)}$ $\left(\cong H_*(\mathbb{P}^{n-1},\mathbb{Q}) \cong A_*(\mathbb{P}^{n-1})\otimes\mathbb{Q}\right)$

Let $X\cdot : V_n \longrightarrow V_n$ be the multiplication by X and

$X^{-1}_\cdot : V_n \longrightarrow V_n$ multiplication by X^{-1}, i.e. $L\circ\dfrac{d}{dX}\circ L^{-1}$

Denote by

$$\boxed{\sigma_+(z)=\sum_{i\geq 0}\sigma_i z^i : \Lambda V_n \longrightarrow \Lambda V_n[[z]]} \quad \& \quad \boxed{\sigma_-(z)=\sum_{i\geq 0}\sigma_{-i} z^{-i} : \Lambda V_n \longrightarrow \Lambda V_n[z^{-1}]}$$

the unique HS-derivations on ΛV_n such that:

$$\sigma_i X^j = X^{i+j} \qquad \sigma_{-i}X^j = (j-i)!\frac{d^i}{dX^i}\left(\frac{X^j}{j!}\right) = \text{"}X^{j-i}\text{"} \quad \boxed{\text{multiplication by } X^{-i}}$$

Their inverses $\overline{\sigma}_+(z)$ and $\overline{\sigma}_-(z)$ are the unique HS-derivations

such that $\boxed{\overline{\sigma}_+(z)X^j = X^j - X^{j+1}z}$ & $\overline{\sigma}_-(z)X^j = X^j - \dfrac{X^{j-1}}{z}$

$X^{2+2} \wedge X^{1+2} \wedge X^0$

Example

$$\sigma_2 \underline{X}^3(2,2) = \sigma_2(X^4 \wedge X^3 \wedge X^0) = \sigma_2(X^4 \wedge X^3) \wedge X^0 + \sigma_1(X^4 \wedge X^3) \wedge \sigma_1 X^0 + X^4 \wedge X^2 \wedge \sigma_2 X^0$$

$$= (\sigma_2 X^4 \wedge X^3 + \sigma_1 X^4 \wedge \sigma_1 X^3 + \cancel{X}^4 \wedge \sigma_2 X^3) \wedge X^0 + (\sigma_1 X^4 \wedge X^3 + X^4 \wedge \sigma_1 X^3) \wedge \sigma_1 X^0 + X^4 \wedge \cancel{X^2 \wedge X^2}$$

$$(X^6 \wedge X^3 + X^5 \cancel{\wedge X^4} + X^4 \cancel{\wedge X^5}) \wedge X^0 + (X^5 \wedge X^3 + X^4 \wedge X^4) \wedge X^1$$

$$X^6 \wedge X^3 \wedge X^0 + \underline{X^5 \wedge X^3 \wedge X^1} = \underline{X}^3(4,2) + \underline{X}^3(3,2,1)$$

So

$$\boxed{\sigma_2 X^3(2,2) = X^3(4,2) + X^3(3,2,1)}$$

Remark $\quad (\Lambda^k V_m)_w = \bigoplus_{|\underset{\sim}{\lambda}|=w} \mathbb{Q}\cdot \underline{X}^k(\underset{\sim}{\lambda}) \qquad (\Lambda^k V_m)_0 = \mathbb{Q}\cdot \underline{X}^k(0) = \mathbb{Q}\cdot X^{k-1}\wedge \dots \wedge X^0$

$$\Big\downarrow \sigma_1^{k(m-k)}$$

$\sigma_1 : (\Lambda^k V_m)_w \longrightarrow (\Lambda^k V_m)_{w+1} \qquad (\Lambda^k V_m)_{k(m-k)} = \mathbb{Q}\cdot \underline{X}^k((m-k)^k) = X^{m-1}\wedge \dots \wedge X^{m-k}$

One more example

$$X^2 \wedge X^1 = X^2 \wedge \sigma_1 X^0 = \sigma_1 (X^2 \wedge X^0) - \sigma_1 X^2 \wedge X^0$$

$$= \sigma_1^2 (\underbrace{X^1 \wedge X^0}) - \underbrace{X^3 \wedge X^0}$$

$$= \sigma_1^2 (X^1 \wedge X^0) - \underline{\sigma_2} (X^1 \wedge X^0)$$

$$= (\sigma_1^2 - \sigma_2)\, X^1 \wedge X^0$$

$$= \begin{vmatrix} \sigma_1 & 1 \\ \sigma_2 & \sigma_1 \end{vmatrix} X^1 \wedge X^0$$

$\dim (\Lambda^k V_m)_0 = 1$

$\dim (\Lambda^k V_m)_{k(m-k)} = 1$

Integration by parts.

Giambelli Formula

$\boxed{\text{Theorem(s)}}$

1) Pieri's Formula holds (–, Asian J. Math, 2005)

$$\boxed{\sigma_i \underline{X}^r(\underline{\lambda}) = \sum_{\underline{\mu}} \underline{X}^r(\underline{\mu})}$$

The sum over all $\mu \in P_r$ / $\mu_1 \geq \lambda_1 \geq \dots \geq \mu_r \geq \lambda_r$ and $|\underline{\mu}| = |\underline{\lambda}| + i$

2) Define $h_i \cdot \underline{X}^r(\underline{\lambda}) := \sigma_i \underline{X}^r(\underline{\lambda})$. Then $\Lambda^r V$ gets a structure of free B_r-module of rank 1 generated by

$$\boxed{X^{n-1} \wedge X^{n-2} \wedge \dots \wedge X^0}$$

such that

$$X^r(\underline{\lambda}) = \Delta_{\underline{\lambda}}(H_r) \cdot X^r(\underline{0})$$

$$= \det(h_{\lambda_j - j + i})$$

Giambelli's Formula

B_r-module

$$\begin{tikzcd}
B_r \arrow[r, "\sim"] \arrow[d] \arrow[dr] & \Lambda^r \mathbb{Q}[X] \arrow[d, shift left] \arrow[d, shift right] \\
B_{r,m} \arrow[r, "\sim"] & \Lambda^r V_m
\end{tikzcd}$$

$B_{r,m} \cong H^*(G(r,m), \mathbb{Q})$

$\Lambda^r V_m \cong H_*(G(r,m), \mathbb{Q})$

$\dfrac{\mathbb{Q}[X}{(X^n,}$

Example:

$\Lambda^2 V_{n+2} \cong H_*(G(2, n+2), \mathbb{Q})$

$\underline{X^1 \wedge X^0} \cong [G(2, n+1)]$ (the fundamental class)

$h_1 \in B_{r,n}$

$\uparrow$

class of a hyperplane in $\mathbb{P}(\Lambda^2 V_{n+1})$

$\cup$

$(\dim)\, G(2, n+1)\, (= 2n)$

$\underline{X^n \wedge X^{n+1}} \cong$ class of a point (codimension $2n$)

$h_1^{2n}[G(2, n+1)] = \deg(G) \cdot [pt]$

$$h_1^{2n}[G(2, n+2)] = \sigma_1^{2n}(X^1 \wedge X^0) = \sum_{j=0}^{2n} \binom{2n}{j} X^{1+j} \wedge X^{2n-j} =$$

$$= \binom{2n}{n} - \binom{2n}{n-1} = \frac{2n!}{n!(n+1)!} = C_n$$

Catalan $\downarrow$

Towards Vertex Operators

$$\begin{tikzcd} \Lambda^r V \arrow[r, "X^i \wedge"] \arrow[d, phantom, "\|"] & \Lambda^{r+1} V \arrow[d, phantom, "\|"] \\ B_r \arrow[r, "? ?"] & B_{r+1} \end{tikzcd}$$

Combinatorial problem

$$\begin{array}{ccc} B_r & & B_{r+1}\cdot X^r(0) \\ \Psi & & \Psi \\ \Delta_\lambda(H_r) & \longrightarrow & X^i \wedge X^r(\lambda) \in B_{r+1} \\ & & X^r(0) \end{array}$$

Find a formula for

The way to do is to consider generating functions

PROPOSITION VO1

$$\sum_{i\geq 1} X^i \cdot z^i \wedge X^r(\lambda) = \sigma_+(z)\, X^0 \wedge X^r(\lambda) = \boxed{z^r \sigma_+(z)\,\sigma_-(z)\, X^{r+1}(\lambda)} \in \Lambda^{r+1} V_m[z^{-1}, z]]$$

If $\partial^j := \frac{1}{j!}\frac{\partial^j}{\partial x^j}\Big|_{x=0} \Rightarrow \partial^j(x^i) = \delta^{ij}$ one has:

$$\sum_j \partial_j w^{-j} \lrcorner X^r(\lambda) = w^{-r+1}\, \bar{\sigma}_+(w)\left(\beta_0 \lrcorner \bar{\sigma}_-(w) X^r(\lambda)\right) \in \Lambda^{r-1} V_m[w^{-1}, w]]$$

PROOF OF PROPOSITION V01 (for $r=1$)

$$\sum_{i\geq 0} X^i z^i \wedge \underline{X}^r(\lambda) = \sum_{i\geq 0} X^i z^i \wedge X^\lambda =$$

$$= \sigma_+(z) X^0 \wedge X^\lambda \qquad \text{(definition of } \sigma_+(z)\text{)}$$

$$\boxed{= \sigma_+(z)\left(X^0 \wedge \bar{\sigma}_+(z) X^\lambda\right)} \qquad \text{(integration by parts)}$$

$$= \sigma_+(z)\left(X^0 \wedge \left(X^\lambda - z.X^{\lambda+1}\right)\right) \qquad \text{(def. of } \bar{\sigma}_+(z)\text{)}$$

$$= z\,\sigma_+(z)\left(\left(X^{\lambda+1} - \frac{X^\lambda}{z}\right) \wedge X^0\right)$$

$$= z\,\sigma_+(z)\left(\bar{\sigma}_-(z) X^{\lambda+1} \wedge \bar{\sigma}_-(z) X^0\right) \qquad \text{(def. } \bar{\sigma}_-(z)\text{)}$$

$$= z\,\sigma_+(z)\,\bar{\sigma}_-(z)\, X^{\lambda+1} \wedge X^0 = z\,\sigma_+(z)\,\bar{\sigma}_-(z)\, X^2(\underline{\lambda}) \qquad \blacksquare$$

In particular

$$\frac{\sum_{i\geq 0} X^i z^i \wedge \underline{X}^r(\underline{\lambda})}{X^r(0)} = \frac{z^r \sigma_+(z)\, \bar{\sigma}_-(z)\, X^{r+1}(\underline{\lambda})}{X^r(0)} = z^r \frac{1}{E_r(z)} \bar{\sigma}_-(z)\, \Delta_\lambda(H_{r+1})$$

$$\boxed{\text{Why do we care ?}}$$

$\mathbb{Q}(X)$

$\shortparallel$

If $r=\infty$, replacing $\Lambda^{\text{``}\infty\text{''}} V$ with $\mathcal{F}(V) = \text{``}\Lambda^{\infty/2} V\text{''}$

vertex operator

then $B = \mathbb{Q}[p_1, p_2, p_3, \dots]$

$\Gamma(z)$

$\shortparallel$

and

$$\frac{1}{E_r(z)} \bar{\sigma}_-(z) \xrightarrow[r\to\infty]{} \boxed{\exp\left(\sum_{i\geq 1} \frac{1}{i} p_i z^i\right) \exp\left(-\sum \frac{1}{z^i} \frac{\partial}{\partial p_i}\right)}$$

Recent results

- Studying $\Lambda\, \mathbb{Q}[X]$ as a representation of $gl_0(\mathbb{Q}[X])$ through suitable vertex operators on Exterior Algebras (Behzad, Contiero, – Vidal Martins, Collectanea Math. 2021). This generalizes (–, Salehyan, Comm. Alg. 2020: "The Cohomology of Grassmannians is a gl_n-module.")

- Studying the $\overset{\displaystyle B}{\cup}$ $\boxed{\text{bosonic}}$ & $\overset{\displaystyle \mathcal{F}}{\cup}$ $\boxed{\text{Fermionic}}$ Representation of $gl(\Lambda\, \mathbb{Q}[X])$ through product of vertex operators, generalizing (–, Salehyan, BBMS, 2020) and Date-Jimbo-Kashiwara-Miwa (1981)

  (Behzad, –, Fundamenta Math. 2021, to appear
  See ArXiv: 2009.00479)

GRAZIE

Probably **Pure Mathematics** is

too difficult for me

but surely

IS NOT REDUNDANT